The Hassenpflug Matrix Tensor Notation

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Abstract

This is a sample document to illustrate the typesetting of vectors, matrices and tensors according to the matrix tensor notation of Hassenpflug (1993, 1995). The first section describes the bare basics of the notation and please note that there is much more to the notation than the little bit described here.

Keywords: vector, matrix, tensor, notation.

N.B.: *This document is neither a guide nor a reference document for the Hassenpflug notation. For any reference to the material in section §1, please cite the original copyrighted articles (Hassenpflug 1993, 1995).*

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1 Hassenpflug matrix tensor notation

1.1 Basic vector notation

All vectors are in the 3-dimensional Euclidean space $\mathbb{R}^3$ and tensors in $\mathbb{R}^{3\times 3}$. Any other vector space will be explicitly stated. The rest of this section lists the basic definitions of the notation of Hassenpflug (1993, 1995)

Physical vector:
$$\vec{\boldsymbol{x}} \equiv \vec{\boldsymbol{e}}_1\, x_1 + \vec{\boldsymbol{e}}_2\, x_2 + \vec{\boldsymbol{e}}_3\, x_3 \tag{1.1}$$

The physical vector is the general representation of a vector in any coordinate system. The unit vectors $\vec{\boldsymbol{e}}_i$, $(i = 1, 2, 3)$, define the direction of the axes in a right-handed orthogonal Cartesian system. The components, $\vec{\boldsymbol{e}}_i\, x_i$, are the components of the vector and the scalar quantities, x_i, the elements of the vector.

Column vector:
$$\overline{\boldsymbol{x}}^a \equiv \begin{bmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \end{bmatrix} \tag{1.2}$$

The column matrix of the elements of a vector is called a column vector and is the algebraic representation of a vector. The bar above the symbol of the vector indicates a column vector and the superscript (a) the index of the specific coordinate system in which the elements of the vector are expressed.

Row vector:
$$\underline{\boldsymbol{x}}_a \equiv \left[\overline{\boldsymbol{x}}^a\right]^{\mathsf{T}} = \begin{bmatrix} x_{a1} & x_{a2} & x_{a3} \end{bmatrix} \tag{1.3}$$

The row matrix of the elements of a vector is called a row vector. The bar below the symbol of the vector indicates a row vector and the subscript (a) the index of the specific coordinate system in which the elements of the vector are expressed. It is important to note that in general is $[\,\overline{\boldsymbol{x}}^a\,]^{\mathsf{T}} = \underline{\boldsymbol{x}}^{\mathsf{T}}{}_a$ for skew and curved coordinates (see Hassenpflug 1995). The format in equation (1.3) without the transpose sign is only valid in Cartesian coordinates.

Norm:
$$\|\vec{\boldsymbol{x}}\| \equiv x, \tag{1.4a}$$
$$\|\overline{\boldsymbol{x}}\| \equiv x \equiv \sqrt{\underline{\boldsymbol{x}} \cdot \overline{\boldsymbol{x}}} = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{1.4b}$$

The norm of a vector is the algebraic size or length of the vector. The second equation, (1.4b), in element form, is only valid in Cartesian coordinates or Euclidean space.

Scalar, dot or inner product:
$$\vec{\boldsymbol{x}} \bullet \vec{\boldsymbol{u}} \equiv \underline{\vec{\boldsymbol{x}}} \cdot \vec{\boldsymbol{u}} = x\, u \cos\varphi, \tag{1.5a}$$
$$\overline{\boldsymbol{x}} \bullet \overline{\boldsymbol{u}} \equiv \underline{\boldsymbol{x}} \cdot \overline{\boldsymbol{u}} = x_1 u_1 + x_2 u_2 + x_3 u_3 \tag{1.5b}$$

The scalar product of two vectors results in a scalar. The angle φ is the angle in space between $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{u}}$.

Dyad or outer product:
$$\overline{\boldsymbol{x}} \circ \overline{\boldsymbol{u}} \equiv \overline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}} = \begin{bmatrix} x_1u_1 & x_1u_2 & x_1u_3 \\ x_2u_1 & x_2u_2 & x_2u_3 \\ x_3u_1 & x_3u_2 & x_3u_3 \end{bmatrix} \tag{1.6}$$

The dyad or outer product of two vectors results in a square matrix. There exists a well defined algebra for dyads. It is sometimes convenient to handle second-rank Cartesian tensors such as inertia tensors as a linear polynomial of dyads, called a dyadic.

Vector or cross product:
$$\begin{aligned} \vec{\boldsymbol{x}} \times \vec{\boldsymbol{u}} \equiv{}& \left(x_2u_3 - x_3u_2\right)\vec{\boldsymbol{e}}_1 \\ &+ \left(x_3u_1 - x_1u_3\right)\vec{\boldsymbol{e}}_2 \\ &+ \left(x_1u_2 - x_2u_1\right)\vec{\boldsymbol{e}}_3 \end{aligned} \tag{1.7a}$$
$$\|\vec{\boldsymbol{x}} \times \vec{\boldsymbol{u}}\| = x\, u \sin\varphi \tag{1.7b}$$

The cross product of the two vector $\vec{x}$ and $\vec{u}$ results in a vector perpendicular to both $\vec{x}$ and $\vec{u}$. This operation is only defined in 3-dimensional Cartesian space. The angle φ is the angle in space between $\vec{x}$ and $\vec{u}$. The cross product can also be defined in terms of a matrix-vector operation $\bar{x} \times \bar{u} \equiv \underline{\bar{\tilde{x}}} \cdot \bar{u}$

Cross product tensor:
$$\underline{\bar{\tilde{x}}} \equiv \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{1.8}$$

Various identities for the cross product tensor can be verified. These identities will be extensively used throughout this article.

$$\begin{aligned}
&\left[\underline{\bar{\tilde{x}}}\right]^{\mathsf{T}} = -\underline{\bar{\tilde{x}}} && \underline{\bar{\tilde{x}}} \cdot \bar{u} = -\underline{\bar{\tilde{u}}} \cdot \bar{x} \\
&\overline{\overline{\underline{\tilde{\bar{x}} \cdot \bar{u}}}} = \underline{\bar{\tilde{x}}} \cdot \underline{\bar{\tilde{u}}} - \underline{\bar{\tilde{u}}} \cdot \underline{\bar{\tilde{x}}} && \overline{\overline{\underline{\widetilde{\bar{x} + \bar{u}}}}} = \underline{\bar{\tilde{x}}} + \underline{\bar{\tilde{u}}} \\
&\left[\underline{\bar{\tilde{x}}}\right]^{2} = \bar{x} \cdot \underline{x} - x^2\, \underline{\bar{I}} && \left[\underline{\bar{\tilde{x}}}\right]^{3} = -x^2\, \underline{\bar{\tilde{x}}} \\
&\left[\underline{\bar{\tilde{x}}}\right]^{2n} = (-1)^{n-1} \left[\underline{\bar{\tilde{x}}}\right]^{2} && \left[\underline{\bar{\tilde{x}}}\right]^{2n-1} = (-1)^{n-1}\, \underline{\bar{\tilde{x}}}
\end{aligned} \tag{1.9}$$

with $\underline{\bar{I}}$ the 3×3 identity matrix.

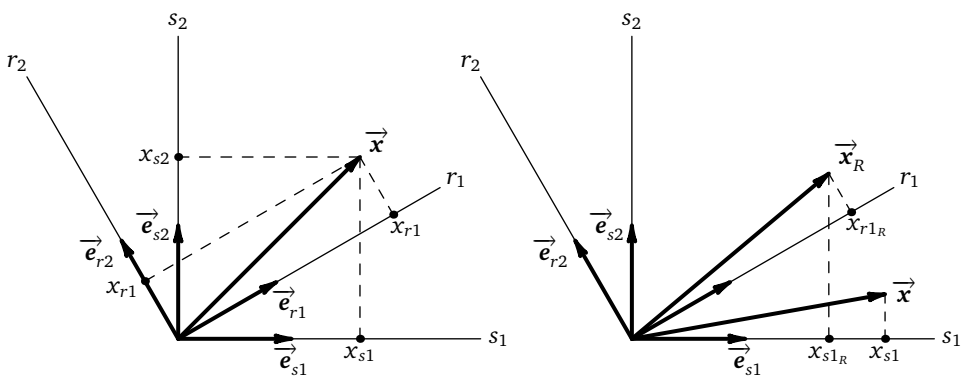
Identity matrix:
$$\underline{\bar{I}} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.10}$$

1.2 Vector transformations

In this section only a basic overview of vector rotations and transformations is given to establish the basic nomenclature and definitions. For a more in-depth discussion refer to Hassenpflug (1993).

Consider two Cartesian axis systems denoted by s and r as shown in figure 1(a). From the general definition of a vector, equation (1.1), it follows

$$\vec{x} = \left[\vec{e}_{s1} \quad \vec{e}_{s2} \quad \vec{e}_{s3}\right] \cdot \begin{bmatrix} x_{s1} \\ x_{s2} \\ x_{s3} \end{bmatrix} = \underline{\vec{E}}_s \cdot \bar{x}_s \tag{1.11}$$



(a) Transformation (b) Rotations

Figure 1: Vector transformations

The quantity, $\underline{\vec{E}}_s = [\vec{e}_{s1}\ \vec{e}_{s2}\ \vec{e}_{s3}]$, is the base of the axis system denoted by s. It consists of the three orthogonal vectors parallel to the axes. From the outer product, equation (1.6), follows for the inverse of base $\underline{\vec{E}}_s$:

$$\left[\underline{\vec{E}}_s\right]^{\mathsf{T}} \cdot \underline{\vec{E}}_s = \underline{\vec{E}}^s \cdot \underline{\vec{E}}_s = \underline{I} \quad \Rightarrow \quad \left[\underline{\vec{E}}_s\right]^{\mathsf{T}} = \left[\underline{\vec{E}}_s\right]^{-1} = \underline{\vec{E}}^s \tag{1.12}$$

We can repeat the procedure of equation (1.11) for the vector $\vec{x}$ in terms of base $\underline{\vec{E}}_r$. The relationship of the elements of vector $\vec{x}$ in terms of base $\underline{\vec{E}}_s$ and base $\underline{\vec{E}}_r$ is then

$$\vec{x} = \underline{\vec{E}}_r \cdot \overline{x}^r = \underline{\vec{E}}_s \cdot \overline{x}^s \quad \Rightarrow \quad \begin{cases} \overline{x}^s &= \underline{\vec{E}}^s \cdot \underline{\vec{E}}_r = \underline{\overline{E}}^s_r \cdot \overline{x}^r \\ \overline{x}^r &= \underline{\vec{E}}^r \cdot \underline{\vec{E}}_s = \underline{\overline{E}}^r_s \cdot \overline{x}^s \end{cases} \tag{1.13}$$

The matrix quantities $\underline{\overline{E}}^s_r$ and $\underline{\overline{E}}^r_s$ are then the transformation matrices of the components of a vector between the two bases $\underline{\vec{E}}_s$ and $\underline{\vec{E}}_r$. The columns of the transformation matrix $\underline{\overline{E}}^s_r$ are the elements of the unit vector $\vec{e}_{s_i}$ expressed in base $\underline{\vec{E}}_s$ and the rows are the unit vectors $\underline{\vec{e}}^{s_j}$ expressed in base $\underline{\vec{E}}_r$.

$$\underline{\overline{E}}^s_r = \left[\overline{e}^s_{r1} \quad \overline{e}^s_{r2} \quad \overline{e}^s_{r3}\right] = \begin{bmatrix} \underline{e}^{s1}_r \\ \underline{e}^{s2}_r \\ \underline{e}^{s3}_r \end{bmatrix} \tag{1.14}$$

The properties of the transformation matrix are well known, for example

$$\left[\underline{\overline{E}}^s_r\right]^{\mathsf{T}} = \left[\underline{\overline{E}}^s_r\right]^{-1} = \underline{\overline{E}}^r_s \tag{1.15}$$

1.3 Vector rotations

Consider the case of a vector in space with initial position $\vec{x}$. The vector is rotated to a new position in space, $\vec{x}_R$. Define the rotation tensor operation then as

$$\vec{x}_R = \underline{\vec{R}} \cdot \vec{x} \tag{1.16}$$

If the operation is applied to the rotation of all the direction vectors of a base $\underline{\vec{E}}_s$ to a new rotated base $\underline{\vec{E}}_r$, then

$$\underline{\vec{E}}_r = \underline{\vec{R}} \cdot \underline{\vec{E}}_s \tag{1.17}$$

or

$$\underline{\overline{E}}^s_r = \underline{\vec{E}}^s \cdot \underline{\vec{R}} \cdot \underline{\vec{E}}_s = \underline{\overline{R}}^s_s \tag{1.18}$$

With reference to figure 1(b), consider the case of a vector fixed in a rotating base $\underline{\vec{E}}_r$ with initial position $\vec{x}$ and final position after a rotation of $\vec{x}_R$. If the initial orientation of $\underline{\vec{E}}_r$ corresponds with that of $\underline{\vec{E}}_s$ then the numerical values of the components of $\overline{x}^s$ and $\overline{x}^r_R$ are equal or $\overline{x}^s = \overline{x}^r_R$. From the transformation of $\vec{x}_R$ it then follows that

$$\overline{x}^s_R = \underline{\overline{R}}^s_s \cdot \overline{x}^s = \underline{\overline{E}}^s_r \cdot \overline{x}^r_R \tag{1.19}$$

If the rotation matrix is transformed between bases, then with the aid of equation (1.18) follows

$$\underline{\overline{R}}^r_r = \underline{\overline{E}}^r_s \cdot \underline{\overline{R}}^s_s \cdot \underline{\overline{E}}^s_r = \left[\underline{\overline{E}}^r_s \cdot \underline{\overline{E}}^s_r\right] \cdot \underline{\overline{E}}^s_r = \underline{\overline{E}}^s_r = \underline{\overline{R}}^s_s \tag{1.20}$$

The rotation matrix is therefore identical in terms of both bases and we can denote it without the base indices if there is no ambiguity. The rotation matrix between bases $\underline{\vec{E}}_s$ and $\underline{\vec{E}}_r$ in terms of the transformation matrix is given by

$$\underline{\overline{R}} = \underline{\overline{E}}_r^s \tag{1.21}$$

$$\left[\underline{\overline{R}}\right]^{-1} = \left[\underline{\overline{R}}\right]^{\mathsf{T}} = \underline{\overline{E}}_s^r \tag{1.22}$$

2 General rotations

2.1 The general rotation matrix (Rodriguez formula)

Euler's theorem states that the most general displacement of a rigid body with one point fixed is equivalent to a single rotation about some axis through that point. With reference to figure 2, consider a vector with initial position $\vec{x}$. The vector is rotated about an axis defined by the unit vector $\vec{a}$, through an angle ϑ. The vector after rotation is denoted by $\vec{x}_R$. From the geometry in figure 2(a) it can be shown (e.g., Shabana 1998, §2.1) for the vector components in terms of the stationary base $\underline{\vec{E}}_s$ that

$$\overline{x}_R^s = \overline{x}^s + \sin\vartheta\,(\overline{a}^s \times \overline{x}^s) + (1-\cos\vartheta)\left(\overline{a}^s \times (\overline{a}^s \times \overline{x}^s)\right) \tag{2.1}$$

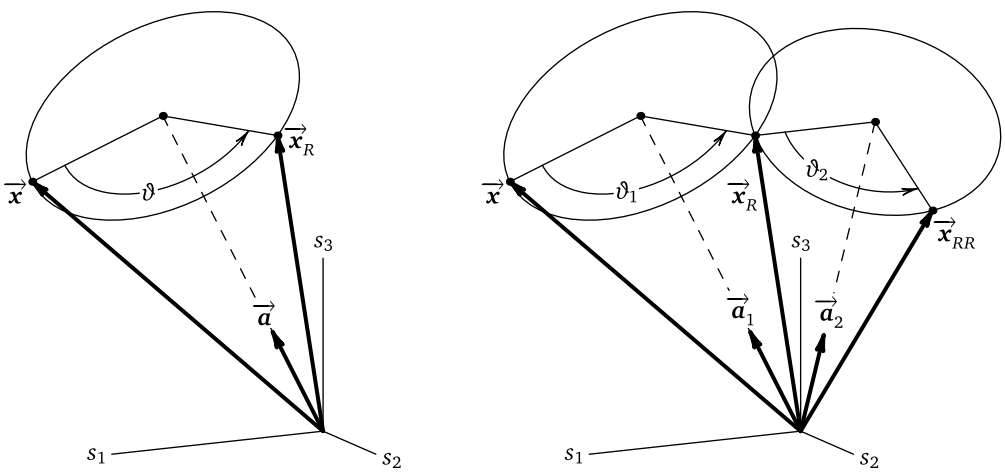
Rewrite equation (2.1) in terms of the cross product tensor defined in equation (1.8)

$$\overline{x}_R^s = \left[\underline{\overline{I}} + \sin\vartheta\,\underline{\overline{\tilde{a}}}_s^s + (1-\cos\vartheta)\,\underline{\overline{\tilde{a}}}_s^s \cdot \underline{\overline{\tilde{a}}}_s^s\right] \cdot \overline{x}^s \tag{2.2}$$

with $\underline{\overline{I}}$ is the 3×3 unit matrix. By comparing equations (2.2) and (1.20), the general format of the rotation matrix for a rotation through an angle ϑ about an axis $\overline{a}^s$ fixed in base $\underline{\vec{E}}_s$ is given by

$$\underline{\overline{R}} = \underline{\overline{I}} + \sin\vartheta\,\underline{\overline{\tilde{a}}}_s^s + (1-\cos\vartheta)\,\underline{\overline{\tilde{a}}}_s^s \cdot \underline{\overline{\tilde{a}}}_s^s \tag{2.3}$$

$$\left[\underline{\overline{R}}\right]^{\mathsf{T}} = \underline{\overline{I}} - \sin\vartheta\,\underline{\overline{\tilde{a}}}_s^s + (1-\cos\vartheta)\,\underline{\overline{\tilde{a}}}_s^s \cdot \underline{\overline{\tilde{a}}}_s^s \tag{2.4}$$



(a) Single rotation

(b) Multiple rotations

Figure 2: General vector rotation

Equation (2.3) is also known as the *Rodriguez formula*. Note that four scalar parameters (ϑ and the three components of $\vec{\boldsymbol{a}}$) and the constraint $\|\vec{\boldsymbol{a}}\|=1$ describe three degrees of rotational freedom.

If $\vec{\boldsymbol{x}}$ is fixed to a rotating base $\vec{\underline{\boldsymbol{E}}}_r$, with $\overline{\boldsymbol{x}}^s = \overline{\boldsymbol{x}}_R^r$ (see figure 1(b)), then $\overline{\underline{\boldsymbol{E}}}_r^s$ is the transformation matrix from base $\vec{\underline{\boldsymbol{E}}}_r$ to base $\vec{\underline{\boldsymbol{E}}}_s$ and

$$\overline{\underline{\boldsymbol{E}}}_r^s = \overline{\underline{\boldsymbol{R}}} \qquad \text{and} \qquad \overline{\underline{\boldsymbol{E}}}_s^r = \left[\overline{\underline{\boldsymbol{R}}}\right]^{\mathsf{T}} \tag{2.5}$$

Note for the transformation of the cross product tensor associated with the rotation axis, is $\overline{\underline{\tilde{\boldsymbol{a}}}}_s^s = \overline{\underline{\tilde{\boldsymbol{a}}}}_r^r = \overline{\underline{\tilde{\boldsymbol{a}}}}$, because the components are identical in both the bases.

Equation (2.3) can also be written in exponential format by expanding $\sin\vartheta$ and $\cos\vartheta$ as Taylor series

$$\begin{aligned} \sin\vartheta &= \vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} + \cdots \\ \cos\vartheta &= 1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} + \cdots \end{aligned} \tag{2.6}$$

With the aid of equation (1.9) follows the elegant solution by Argyris (1982)

$$\begin{aligned} \overline{\underline{\boldsymbol{R}}} &= \overline{\underline{\boldsymbol{I}}} + \left(\vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} + \cdots\right)\overline{\underline{\tilde{\boldsymbol{a}}}} + \left(\frac{\vartheta^2}{2!} - \frac{\vartheta^4}{4!} + \cdots\right)\left[\overline{\underline{\tilde{\boldsymbol{a}}}}\right]^2 \\ &= \overline{\underline{\boldsymbol{I}}} + \vartheta\,\overline{\underline{\tilde{\boldsymbol{a}}}} + \frac{\vartheta^2}{2!}\left[\overline{\underline{\tilde{\boldsymbol{a}}}}\right]^2 + \frac{\vartheta^3}{3!}\left[\overline{\underline{\tilde{\boldsymbol{a}}}}\right]^3 + \cdots + \frac{\vartheta^n}{n!}\left[\overline{\underline{\tilde{\boldsymbol{a}}}}\right]^n + \cdots \end{aligned} \tag{2.7}$$

which is the exponential matrix

$$\overline{\underline{\boldsymbol{R}}} = \mathrm{e}^{\vartheta\overline{\underline{\tilde{\boldsymbol{a}}}}} \qquad \text{and} \qquad \left[\overline{\underline{\boldsymbol{R}}}\right]^{\mathsf{T}} = \mathrm{e}^{-\vartheta\overline{\underline{\tilde{\boldsymbol{a}}}}} \tag{2.8}$$

For numerical purposes equation (2.3) can be written as a single matrix. Let $c = \cos\vartheta$ and $s = \sin\vartheta$, then the rotation or transformation matrix is given by

$$\overline{\underline{\boldsymbol{R}}} = \overline{\underline{\boldsymbol{E}}}_r^s = \begin{bmatrix} a_1^2(1-c)+c & a_1a_2(1-c)-a_3s & a_1a_3(1-c)+a_2s \\ a_1a_2(1-c)+a_3s & a_2^2(1-c)+c & a_2a_3(1-c)-a_1s \\ a_1a_3(1-c)-a_2s & a_2a_3(1-c)+a_1s & a_3^2(1-c)+c \end{bmatrix} \tag{2.9}$$

It is frequently necessary to find the rotation axis $\overline{\boldsymbol{a}}$ and rotation angle ϑ for a known transformation matrix, $\overline{\underline{\boldsymbol{E}}}_r^s = [\,E_{ij}\,]$. From equation (2.9) various relationships can be deducted. Two of the more important ones are

$$2\cos\vartheta = E_{11} + E_{22} + E_{33} - 1 \tag{2.10}$$

$$2\sin\vartheta\,\overline{\boldsymbol{a}} = \begin{bmatrix} E_{32} - E_{23} \\ E_{13} - E_{31} \\ E_{21} - E_{12} \end{bmatrix} \tag{2.11}$$

When $\vartheta \approx \pi$ equation (2.11) can not be used to find $\overline{\boldsymbol{a}}$. Another more general approach, is to consider the characteristic polynomial of $\overline{\underline{\boldsymbol{E}}}_r^s$.

$$\det[\,\overline{\underline{\boldsymbol{E}}}_r^s - \lambda\overline{\underline{\boldsymbol{I}}}\,] = (\lambda^2 + 2\lambda\cos\vartheta + 1)(1-\lambda) = 0 \tag{2.12}$$

It leads to the eigenvalues $\lambda = \mathrm{e}^{\mathrm{i}\vartheta}$, $\mathrm{e}^{-\mathrm{i}\vartheta}$, 1. It can therefore be stated that $\lambda = 1$ is always an eigenvalue of $\overline{\underline{\boldsymbol{E}}}_r^s$ and that an eigenvector or axis $\overline{\boldsymbol{a}} = \overline{\boldsymbol{a}}^s = \overline{\boldsymbol{a}}^r$ exists that is unchanged by the rotation. The rotation axis can be obtained with a numerical method by solving the eigenvector problem $\overline{\underline{\boldsymbol{E}}}_r^s \cdot \overline{\boldsymbol{a}} = \overline{\boldsymbol{a}}$.

2.2 Multiple rotations

For the case of multiple rotations of a vector as shown in figure 2(b), let

$$\overline{\boldsymbol{x}}_R^s = \underline{\overline{\boldsymbol{R}_1}} \cdot \overline{\boldsymbol{x}}^s \qquad \text{with} \qquad \underline{\overline{\boldsymbol{R}_1}} = \underline{\overline{\boldsymbol{R}}}(\vartheta_1, \overrightarrow{\boldsymbol{a}}_1) \tag{2.13}$$

$$\overline{\boldsymbol{x}}_{RR}^s = \underline{\overline{\boldsymbol{R}_2}} \cdot \overline{\boldsymbol{x}}_R^s \qquad \text{with} \qquad \underline{\overline{\boldsymbol{R}_2}} = \underline{\overline{\boldsymbol{R}}}(\vartheta_2, \overrightarrow{\boldsymbol{a}}_2) \tag{2.14}$$

then

$$\overline{\boldsymbol{x}}_{RR}^s = \underline{\overline{\boldsymbol{R}_2}} \cdot \overline{\boldsymbol{x}}_R^s = \underline{\overline{\boldsymbol{R}_2}} \cdot \underline{\overline{\boldsymbol{R}_1}} \cdot \overline{\boldsymbol{x}}^s = \underline{\overline{\boldsymbol{R}}} \cdot \overline{\boldsymbol{x}}^s \tag{2.15}$$

with

$$\underline{\overline{\boldsymbol{R}}} = \underline{\overline{\boldsymbol{R}_2}} \cdot \underline{\overline{\boldsymbol{R}_1}} \tag{2.16}$$

If $\overrightarrow{\boldsymbol{x}}$ is fixed to a rotating base $\overrightarrow{\underline{\boldsymbol{E}}}_r$, with $\overline{\boldsymbol{x}}^s = \overline{\boldsymbol{x}}_{RR}^r$, then $\underline{\overline{\boldsymbol{E}}}_r^s$ is the transformation matrix from base $\overrightarrow{\underline{\boldsymbol{E}}}_r$ to base $\overrightarrow{\underline{\boldsymbol{E}}}_s$ and

$$\underline{\overline{\boldsymbol{E}}}_r^s = \underline{\overline{\boldsymbol{R}_2}} \cdot \underline{\overline{\boldsymbol{R}_1}} \tag{2.17}$$

$$\underline{\overline{\boldsymbol{E}}}_s^r = \left[\underline{\overline{\boldsymbol{R}_2}} \cdot \underline{\overline{\boldsymbol{R}_1}}\right]^{\mathsf{T}} = \left[\underline{\overline{\boldsymbol{R}_1}}\right]^{\mathsf{T}} \cdot \left[\underline{\overline{\boldsymbol{R}_2}}\right]^{\mathsf{T}} \tag{2.18}$$

Note that in general is $\underline{\overline{\boldsymbol{R}_2}} \cdot \underline{\overline{\boldsymbol{R}_1}} \neq \underline{\overline{\boldsymbol{R}_2}} \cdot \underline{\overline{\boldsymbol{R}_1}}$. If we write equation (2.18) in terms of the exponential representation of equation (2.8) then

$$\underline{\overline{\boldsymbol{R}}} = \mathbf{e}^{\vartheta_1 \underline{\overline{\widetilde{\boldsymbol{a}_1}}}} \cdot \mathbf{e}^{\vartheta_2 \underline{\overline{\widetilde{\boldsymbol{a}_2}}}} \neq \mathbf{e}^{\left(\vartheta_1 \underline{\overline{\widetilde{\boldsymbol{a}_1}}} + \vartheta_2 \underline{\overline{\widetilde{\boldsymbol{a}_2}}}\right)} \tag{2.19}$$

This means that the rotations are not vectors that can be added. The only exception is when the rotation axes are parallel, $\overline{\boldsymbol{a}}_1 \parallel \overline{\boldsymbol{a}}_2$.

2.3 Infinitesimal rotations

In case of an infinitesimal rotation $\Delta\vartheta$, second and higher order terms in the series expansion in equation (2.7) can be neglected, resulting in

$$\Delta\underline{\overline{\boldsymbol{R}}} \approx \underline{\overline{\boldsymbol{I}}} + \Delta\vartheta\, \underline{\overline{\widetilde{\boldsymbol{a}}}} \qquad \text{and} \qquad \left[\Delta\underline{\overline{\boldsymbol{R}}}\right]^{\mathsf{T}} \approx \underline{\overline{\boldsymbol{I}}} - \Delta\vartheta\, \underline{\overline{\widetilde{\boldsymbol{a}}}} \tag{2.20}$$

In the previous section it was proven that finite rotations are not vector quantities that can be added. Infinitesimal rotation are vector quantities that can be added to give a total rotation. Consider two infinitesimal rotations $\Delta\vartheta_1$ and $\Delta\vartheta_2$ about axes $\overline{\boldsymbol{a}}_1$ and $\overline{\boldsymbol{a}}_2$

$$\Delta\underline{\overline{\boldsymbol{R}_1}} = \underline{\overline{\boldsymbol{I}}} + \Delta\vartheta_1\, \underline{\overline{\widetilde{\boldsymbol{a}_1}}} \qquad \text{and} \qquad \Delta\underline{\overline{\boldsymbol{R}_2}} = \underline{\overline{\boldsymbol{I}}} + \Delta\vartheta_2\, \underline{\overline{\widetilde{\boldsymbol{a}_2}}} \tag{2.21}$$

For a multiple rotation

$$\begin{aligned}
\Delta\underline{\overline{\boldsymbol{R}_1}} \cdot \Delta\underline{\overline{\boldsymbol{R}_2}} &= \left[\underline{\overline{\boldsymbol{I}}} + \Delta\vartheta_1\, \underline{\overline{\widetilde{\boldsymbol{a}_1}}}\right] \cdot \left[\underline{\overline{\boldsymbol{I}}} + \Delta\vartheta_2\, \underline{\overline{\widetilde{\boldsymbol{a}_2}}}\right] \\
&= \underline{\overline{\boldsymbol{I}}} + \Delta\vartheta_1\, \underline{\overline{\widetilde{\boldsymbol{a}_1}}} + \Delta\vartheta_2\, \underline{\overline{\widetilde{\boldsymbol{a}_2}}} + \Delta\vartheta_1\, \Delta\vartheta_2\, \underline{\overline{\widetilde{\boldsymbol{a}_1}}} \cdot \underline{\overline{\widetilde{\boldsymbol{a}_2}}} \\
&\approx \underline{\overline{\boldsymbol{I}}} + \Delta\vartheta_1\, \underline{\overline{\widetilde{\boldsymbol{a}_1}}} + \Delta\vartheta_2\, \underline{\overline{\widetilde{\boldsymbol{a}_2}}}
\end{aligned} \tag{2.22}$$

where second and higher order terms were again ignored. This results in

$$\Delta\underline{\overline{\boldsymbol{R}_1}} \cdot \Delta\underline{\overline{\boldsymbol{R}_2}} = \Delta\underline{\overline{\boldsymbol{R}_2}} \cdot \Delta\underline{\overline{\boldsymbol{R}_1}} \tag{2.23}$$

proving that two successive infinitesimal rotations about different axes can be added and that an infinitesimal rotation is a vector. For n successive rotations it can be shown that

$$\begin{aligned}\Delta\underline{\overline{R_1}}\cdot\Delta\underline{\overline{R_2}}\cdots\Delta\underline{\overline{R_n}} &= \prod_{i=1}^{n}\Delta\underline{\overline{R_i}} = \underline{\overline{I}} + \Delta\vartheta_1\,\underline{\overline{\widetilde{a_1}}} + \Delta\vartheta_2\,\underline{\overline{\widetilde{a_2}}} + \cdots + \Delta\vartheta_n\,\underline{\overline{\widetilde{a_n}}} \\ &= \underline{\overline{I}} + \sum_{i=1}^{n}\Delta\vartheta_i\,\underline{\overline{\widetilde{a_i}}} \\ &= \Delta\underline{\overline{R_n}}\cdot\Delta\underline{\overline{R_{n-1}}}\cdots\Delta\underline{\overline{R_1}}\end{aligned} \tag{2.24}$$

3 Rotation kinematics

3.1 Angular velocity

Consider three successive infinitesimal rotations $\Delta\vartheta_1$, $\Delta\vartheta_2$ and $\Delta\vartheta_3$ about the unit vectors in the axis directions $\overline{e}_1^s=[1\quad 0\quad 0]^\mathsf{T}$, $\overline{e}_2^s=[0\quad 1\quad 0]^\mathsf{T}$ and $\overline{e}_3^s=[0\quad 0\quad 1]^\mathsf{T}$. The total infinitesimal rotation is then from equation (2.24)

$$\Delta\underline{\overline{R}} = \Delta\underline{\overline{R_1}}\cdot\Delta\underline{\overline{R_2}}\cdot\Delta\underline{\overline{R_3}} = \underline{\overline{I}} + \Delta\underline{\overline{\vartheta}}_s^s \tag{3.1}$$

with

$$\Delta\underline{\overline{\vartheta}}_s^s = \Delta\vartheta_1\,\underline{\overline{\widetilde{e_1}}}_s^s + \Delta\vartheta_2\,\underline{\overline{\widetilde{e_2}}}_s^s + \Delta\vartheta_3\,\underline{\overline{\widetilde{e_3}}}_s^s = \begin{bmatrix} 0 & -\Delta\vartheta_3 & \Delta\vartheta_2 \\ \Delta\vartheta_3 & 0 & -\Delta\vartheta_1 \\ -\Delta\vartheta_2 & \Delta\vartheta_1 & 0 \end{bmatrix} \tag{3.2}$$

The total infinitesimal rotation of a vector $\overline{x}^s$ with fixed length about three perpendicular axes is then

$$\overline{x}_\Delta^s = \Delta\underline{\overline{R}}\cdot\overline{x}^s = \left[\underline{\overline{I}} + \Delta\underline{\overline{\vartheta}}_s^s\right]\cdot\overline{x}^s = \overline{x}^s + \Delta\underline{\overline{\vartheta}}_s^s\cdot\overline{x}^s \tag{3.3}$$

The change vector is

$$\Delta\overline{x}^s = \overline{x}_\Delta^s - \overline{x}^s = \Delta\underline{\overline{\vartheta}}_s^s\cdot\overline{x}^s \tag{3.4}$$

Divide equation (3.4) by the time increment Δt during which the rotations take place. For the limit as Δt approaches zero follows

$$\lim_{\Delta t\to 0}\frac{\Delta\overline{x}^s}{\Delta t} = \frac{\mathrm{d}\overline{x}^s}{\mathrm{d}t} = \dot{\overline{x}}^s \tag{3.5}$$

and

$$\lim_{\Delta t\to 0}\frac{\Delta\underline{\overline{\vartheta}}_s^s}{\Delta t}\cdot\overline{x}^s = \underline{\overline{\widetilde{\omega}}}_s^s\cdot\overline{x}^s = \overline{\omega}^s\times\overline{x}^s \tag{3.6}$$

so that the time derivative of a rotating vector of fixed length become

$$\dot{\overline{x}}^s = \underline{\overline{\widetilde{\omega}}}_s^s\cdot\overline{x}^s = \overline{\omega}^s\times\overline{x}^s \tag{3.7}$$

The vector $\overrightarrow{\omega}$ is defined as the angular velocity with components

$$\omega_i = \lim_{\Delta t\to 0}\frac{\Delta\vartheta_i}{\Delta t} \qquad i = 1,2,3 \tag{3.8}$$

the instantaneous rotation rate about the three coordinate axes.

It must emphasized that the angular velocity is a defined vector and *not the derivative of any quantity*. This implies that the angular velocity cannot be integrated to the obtain the attitude or orientations of a vector or base or any other quantity.

As an application of equation (3.7), the time derivative of a transformation matrix in equation (1.14) from a static base $\overrightarrow{\underline{E}}_s$ to a rotating base $\overrightarrow{\underline{E}}_r$ can be obtained

$$\dot{\underline{\overline{E}}}_r^s = \left[\dot{\overline{e}}_{r1}^s \quad \dot{\overline{e}}_{r2}^s \quad \dot{\overline{e}}_{r3}^s\right] = \left[\underline{\overline{\tilde{\omega}}}_s^s \cdot \overline{e}_{r1}^s \quad \underline{\overline{\tilde{\omega}}}_s^s \cdot \overline{e}_{r2}^s \quad \underline{\overline{\tilde{\omega}}}_s^s \cdot \overline{e}_{r3}^s\right] = \underline{\overline{\tilde{\omega}}}_s^s \cdot \underline{\overline{E}}_r^s \tag{3.9}$$

or for the angular velocity in terms of the rotating base $\overrightarrow{\underline{E}}_r$

$$\dot{\underline{\overline{E}}}_r^s = \left[\underline{\overline{E}}_r^s \cdot \underline{\overline{\tilde{\omega}}}_r^r \cdot \underline{\overline{E}}_s^r\right] \cdot \underline{\overline{E}}_r^s = \underline{\overline{E}}_r^s \cdot \underline{\overline{\tilde{\omega}}}_r^r \tag{3.10}$$

3.2 Rotation kinematics

Define the vectors $\overline{\boldsymbol{x}}^s$ and $\dot{\overline{\boldsymbol{x}}}^s = \mathrm{d}\overline{\boldsymbol{x}}^s/\mathrm{d}t$ as the position and velocity of a particle or point with components in terms of a static base $\overrightarrow{\underline{E}}_s$, while $\overline{\boldsymbol{x}}^r$ and $\dot{\overline{\boldsymbol{x}}}^r$ are the position and apparent velocity in terms of a rotating base $\overrightarrow{\underline{E}}_r$.

$$\overline{\boldsymbol{x}}^s = \underline{\overline{E}}_r^s \cdot \overline{\boldsymbol{x}}^r \tag{3.11}$$

and

$$\dot{\overline{\boldsymbol{x}}}^s = \underline{\overline{E}}_r^s \cdot \left[\dot{\overline{\boldsymbol{x}}}^r + \underline{\overline{E}}_s^r \cdot \dot{\underline{\overline{E}}}_r^s \cdot \overline{\boldsymbol{x}}^r\right] = \underline{\overline{E}}_r^s \cdot \left[\dot{\overline{\boldsymbol{x}}}^r + \underline{\overline{\tilde{\omega}}}_r^r \cdot \overline{\boldsymbol{x}}^r\right] \tag{3.12}$$

The cross product tensor of the angular velocity $\overrightarrow{\boldsymbol{\omega}}$ is from equations (3.9) and (3.9)

$$\begin{aligned} \underline{\overline{\tilde{\omega}}}_r^r &= \underline{\overline{E}}_s^r \cdot \dot{\underline{\overline{E}}}_r^s \\ \underline{\overline{\tilde{\omega}}}_s^s &= \underline{\overline{E}}_r^s \cdot \underline{\overline{\tilde{\omega}}}_r^r \cdot \underline{\overline{E}}_s^r = \dot{\underline{\overline{E}}}_s^r \cdot \underline{\overline{E}}_s^r \end{aligned} \tag{3.13}$$

We proceed next to obtain $\overrightarrow{\boldsymbol{\omega}}$ as a function of $\overrightarrow{\boldsymbol{a}}$ and ϑ. The following identities can then be verified from the fact that $\overrightarrow{\boldsymbol{a}}$ is a unit vector, $(\underline{\boldsymbol{a}} \cdot \overline{\boldsymbol{a}} = 1)$, implying that $(\underline{\boldsymbol{a}} \cdot \dot{\overline{\boldsymbol{a}}} = 0)$:

$$\begin{aligned} \underline{\overline{\tilde{a}}} \cdot \dot{\underline{\overline{\tilde{a}}}} \cdot \underline{\overline{\tilde{a}}} &= -(\underline{\boldsymbol{a}} \cdot \dot{\overline{\boldsymbol{a}}})\, \underline{\overline{\tilde{a}}} \quad = \underline{\overline{0}} \\ \underline{\overline{\tilde{a}}} \cdot \underline{\overline{\tilde{a}}} \cdot \dot{\underline{\overline{\tilde{a}}}} \cdot \underline{\overline{\tilde{a}}} &= -(\underline{\boldsymbol{a}} \cdot \dot{\overline{\boldsymbol{a}}})\, \underline{\overline{\tilde{a}}} \cdot \underline{\overline{\tilde{a}}} = \underline{\overline{0}} \end{aligned} \tag{3.14}$$

The angular velocity tensor in equation (3.13), after the differentiation of the transformation matrix equation (2.3) and algebraic manipulation with the aid of equations (3.14) and (1.9) is

$$\begin{aligned} \underline{\overline{\tilde{\omega}}}_r^r &= \dot{\vartheta}\, \underline{\overline{\tilde{a}}} + \sin\vartheta\, \dot{\underline{\overline{\tilde{a}}}} - 2\sin^2\tfrac{\vartheta}{2} \left[\underline{\overline{\tilde{a}}} \cdot \dot{\underline{\overline{\tilde{a}}}} - \dot{\underline{\overline{\tilde{a}}}} \cdot \underline{\overline{\tilde{a}}}\right] \\ &= \dot{\vartheta}\, \underline{\overline{\tilde{a}}} + \sin\vartheta\, \dot{\underline{\overline{\tilde{a}}}} - 2\sin^2\tfrac{\vartheta}{2}\, \underline{\overline{\tilde{\tilde{a}} \cdot \dot{\tilde{a}}}} \end{aligned} \tag{3.15}$$

From equation (3.15), the vector equation for $\overline{\boldsymbol{\omega}}^r$ and $\overline{\boldsymbol{\omega}}^s$ (where the latter can be derived with the same arguments), follows then as

$$\begin{aligned} \overline{\boldsymbol{\omega}}^r &= \dot{\vartheta}\, \overline{\boldsymbol{a}} + \sin\vartheta\, \dot{\overline{\boldsymbol{a}}} - 2\sin^2\tfrac{\vartheta}{2}\, \underline{\overline{\tilde{a}}} \cdot \dot{\overline{\boldsymbol{a}}} \\ \overline{\boldsymbol{\omega}}^s &= \dot{\vartheta}\, \overline{\boldsymbol{a}} + \sin\vartheta\, \dot{\overline{\boldsymbol{a}}} + 2\sin^2\tfrac{\vartheta}{2}\, \underline{\overline{\tilde{a}}} \cdot \dot{\overline{\boldsymbol{a}}} \end{aligned} \tag{3.16}$$

The inner or scalar product of equation (3.16) gives the norm of the angular velocity

$$\omega^2 = \underline{\boldsymbol{\omega}}_r \cdot \overline{\boldsymbol{\omega}}^r = \underline{\boldsymbol{\omega}}_s \cdot \overline{\boldsymbol{\omega}}^s = \dot{\vartheta}^2 + 4\sin\tfrac{\vartheta}{2}\dot{a}^2 \tag{3.17}$$

From equation (3.16) the time derivative of the rotation angle ϑ is

$$\dot{\vartheta} = \underline{\boldsymbol{a}} \cdot \overline{\boldsymbol{\omega}}^r = \underline{\boldsymbol{a}} \cdot \overline{\boldsymbol{\omega}}^s \tag{3.18}$$

Multiply equation (3.18) with $\overline{\boldsymbol{a}}$. With the aid of the triple cross-product identities, it then follows

$$\begin{aligned}\dot{\vartheta}\,\overline{\boldsymbol{a}} &= (\underline{\boldsymbol{a}}\cdot\overline{\boldsymbol{\omega}}^r)\,\overline{\boldsymbol{a}} = \overline{\boldsymbol{\omega}}^r + \underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\overline{\boldsymbol{\omega}}^r \\ &= (\underline{\boldsymbol{a}}\cdot\overline{\boldsymbol{\omega}}^s)\,\overline{\boldsymbol{a}} = \overline{\boldsymbol{\omega}}^s + \underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\overline{\boldsymbol{\omega}}^s\end{aligned} \tag{3.19}$$

Inspection of equations (3.16) to (3.18) reveals that $\underline{\boldsymbol{a}}\cdot\overrightarrow{\boldsymbol{\omega}} = \dot{\vartheta} \neq \omega$. The angular velocity vector $\overrightarrow{\boldsymbol{\omega}}$ is therefore in general not in the direction of the instantaneous rotation axis $\overrightarrow{\boldsymbol{a}}$.

The vector $\dot{\overline{\boldsymbol{a}}}$ can be obtained from equation (3.16) by the substitution of equation (3.19) and assuming a solution of the form $[\,\underline{\overline{\boldsymbol{I}}} + \alpha\,\underline{\overline{\tilde{\boldsymbol{a}}}} + \beta\,\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\,]$. With the aid of the identities in equations (3.14) and (1.9), it leads to

$$\begin{aligned}\dot{\overline{\boldsymbol{a}}} &= \tfrac{1}{2}\left[+\underline{\overline{\tilde{\boldsymbol{a}}}} - \cot\tfrac{\vartheta}{2}\,\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\right]\cdot\overline{\boldsymbol{\omega}}^r \equiv \underline{\overline{\boldsymbol{K}}}_r\cdot\overline{\boldsymbol{\omega}}^r \\ &= \tfrac{1}{2}\left[-\underline{\overline{\tilde{\boldsymbol{a}}}} - \cot\tfrac{\vartheta}{2}\,\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\right]\cdot\overline{\boldsymbol{\omega}}^s \equiv \underline{\overline{\boldsymbol{K}}}_s\cdot\overline{\boldsymbol{\omega}}^s\end{aligned} \tag{3.20}$$

Note the notation in equation (3.20) for $\underline{\overline{\boldsymbol{K}}}_r$. It is a tensor in a mixed base (see Hassenpflug 1993), because $\overline{\boldsymbol{a}}^r = \overline{\boldsymbol{a}}^s$. For the transformation between bases it can also be confirmed that

$$\underline{\overline{\boldsymbol{K}}}_r = \underline{\overline{\boldsymbol{K}}}_s\cdot\underline{\overline{\boldsymbol{E}}}_r^s \tag{3.21}$$

The general kinematic equations for a rotating base are given by equation (3.18) and equation (3.20). The four scalar equations describe only three degrees of freedom and are constrained by $\|\overline{\boldsymbol{a}}\| = 1$. These equations can be integrated to obtain $\underline{\overline{\boldsymbol{E}}}_r^s$ as a functions of time, but equation (3.20) is singular for values of $\vartheta = 0, \pm 2\pi, \cdots$, which render a general numeric solution impractical.

4 Attitude determination

4.1 General

The classic problem in rotation kinematics is that the angular velocity cannot be integrated to obtain the orientation of a rotating base, because the integral is dependent on the path of integration. The most basic method to find the orientation of $\overrightarrow{\underline{\boldsymbol{E}}}_r$ as a function of time is to integrate equation (3.13) directly,

$$\begin{aligned}\dot{\underline{\overline{\boldsymbol{E}}}}_r^s &= \underline{\overline{\tilde{\boldsymbol{\omega}}}}_s^s\cdot\underline{\overline{\boldsymbol{E}}}_r^s = \begin{bmatrix}\overline{\boldsymbol{\omega}}^s\times\overline{\boldsymbol{e}}_{r1}^s & \overline{\boldsymbol{\omega}}^s\times\overline{\boldsymbol{e}}_{r2}^s & \overline{\boldsymbol{\omega}}^s\times\overline{\boldsymbol{e}}_{r3}^s\end{bmatrix} \\ \dot{\underline{\overline{\boldsymbol{E}}}}_s^r &= -\underline{\overline{\tilde{\boldsymbol{\omega}}}}_r^r\cdot\underline{\overline{\boldsymbol{E}}}_s^r = -\begin{bmatrix}\overline{\boldsymbol{\omega}}^r\times\overline{\boldsymbol{e}}_{s1}^r & \overline{\boldsymbol{\omega}}^r\times\overline{\boldsymbol{e}}_{s2}^r & \overline{\boldsymbol{\omega}}^r\times\overline{\boldsymbol{e}}_{s3}^r\end{bmatrix}\end{aligned} \tag{4.1}$$

Only two of the vectors need to be integrated. The third vector can be obtained from the cross product $(\overline{\boldsymbol{e}}_1\times\overline{\boldsymbol{e}}_2 = \overline{\boldsymbol{e}}_3)$. This method involves six parameters while there are only three degrees of freedom. With a lot of effort and by careful selection of elements from the orthogonality constraint requirement $\underline{\overline{\boldsymbol{E}}}_r^s\cdot\underline{\overline{\boldsymbol{E}}}_s^r = \underline{\overline{\boldsymbol{I}}}$, it can be refined to three parameters. It is also advisable that the constraint equation be enforced through frequent normalization, to compensate for the fact that the constraints are not taken into account during integration.

4.2 Euler symmetric parameters

Throughout history many parameterization methods were devised to obtain the relationships between the orientation of a rotating base and its angular velocity.

The Euler symmetric parameter method[1] is one of the classic methods. It has gained popularity in the aerospace engineering environment for foolproof attitude determination algorithms, because it contains no numerical singularities. It has the disadvantage that it is a four-parameter method describing three degrees of freedom, and therefore an additional differential equation, together with its constraint, must be solved.

After inspection of equation (2.3), define the four Euler parameters

$$q_0 = \cos\tfrac{\vartheta}{2} \qquad \overline{\boldsymbol{q}} = \sin\tfrac{\vartheta}{2}\,\overline{\boldsymbol{a}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{4.2}$$

The transformation matrix equation (2.3), in terms of the Euler parameters, is then

$$\underline{\overline{\boldsymbol{E}}}_r^s(q_0, \overline{\boldsymbol{q}}) = \underline{\overline{\boldsymbol{I}}} + 2q_0\,\underline{\overline{\tilde{\boldsymbol{q}}}} + 2\,\underline{\overline{\tilde{\boldsymbol{q}}}} \cdot \underline{\overline{\tilde{\boldsymbol{q}}}} \tag{4.3}$$

$$\underline{\overline{\boldsymbol{E}}}_s^r(q_0, \overline{\boldsymbol{q}}) = \underline{\overline{\boldsymbol{E}}}_r^s(q_0, -\overline{\boldsymbol{q}}) \tag{4.4}$$

or in element form

$$\underline{\overline{\boldsymbol{E}}}_r^s(q_0, \overline{\boldsymbol{q}}) = \begin{bmatrix} 2q_0^2 + 2q_1^2 - 1 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0) \\ 2(q_1q_2 + q_3q_0) & 2q_0^2 + 2q_2^2 - 1 & 2(q_2q_3 - q_1q_0) \\ 2(q_1q_3 - q_2q_0) & 2(q_2q_3 + q_1q_0) & 2q_0^2 + 2q_3^2 - 1 \end{bmatrix} \tag{4.5}$$

The four Euler parameters are not independent, but are constrained by the condition for the transformation matrix, $\underline{\overline{\boldsymbol{E}}}_r^s \cdot \underline{\overline{\boldsymbol{E}}}_s^r = \underline{\overline{\boldsymbol{I}}}$, which implies that

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = q_0^2 + \underline{\boldsymbol{q}} \cdot \overline{\boldsymbol{q}} = 1 \tag{4.6}$$

and which is indeed satisfied by equation (4.2).

From equation (4.3) it is clear that changing the signs of all the Euler parameters simultaneously does not affect the transformation matrix

$$\underline{\overline{\boldsymbol{E}}}_r^s(-q_0, -\overline{\boldsymbol{q}}) = \underline{\overline{\boldsymbol{E}}}_r^s(q_0, \overline{\boldsymbol{q}}) \tag{4.7}$$

The initial values of q_0 and $\overline{\boldsymbol{q}}$ can be obtained for a known transformation matrix $\underline{\overline{\boldsymbol{E}}}_r^s = \left[E_{ij}\right]$ from equation (4.5). The following equations are the relationships that can be deducted

$$\begin{aligned} 4q_0^2 &= 1 + E_{11} + E_{22} + e_{33} \\ 4q_1^2 &= 1 + E_{11} - E_{22} - e_{33} \\ 4q_2^2 &= 1 - E_{11} + E_{22} - E_{33} \\ 4q_3^2 &= 1 - E_{11} - E_{22} + E_{33} \end{aligned} \tag{4.8}$$

$$\begin{aligned} 4q_1q_0 &= E_{32} - E_{23} & & 4q_1q_2 &= E_{12} + E_{21} \\ 4q_2q_0 &= E_{13} - E_{31} & \text{and} \quad & 4q_1q_3 &= E_{13} + E_{31} \\ 4q_3q_0 &= E_{21} - E_{12} & & 4q_2q_3 &= E_{23} + E_{32} \end{aligned} \tag{4.9}$$

The absolute values of Euler parameters are obtained from equation (4.8).

$$\begin{aligned} |2\,q_0| &= \sqrt{1 + E_{11} + E_{22} + E_{33}} \\ |2\,q_1| &= \sqrt{1 + E_{11} - E_{22} - E_{33}} \\ |2\,q_2| &= \sqrt{1 - E_{11} + E_{22} - E_{33}} \\ |2\,q_3| &= \sqrt{1 - E_{11} - E_{22} + E_{33}} \end{aligned} \tag{4.10}$$

[1] It is also called the rotation quaternion because it can be represented as a unit quaternion, obeying all the rules of quaternion algebra.

The unity constraint equation (4.6), implies that at least one of the Euler parameters is not zero. Furthermore, a simultaneous sign change of all the Euler parameters has no effect on the transformation matrix, see equation (4.7). To avoid singularities and for the best numerical accuracy, select the absolute value of the largest parameter from equation (4.10) as initial value and then calculate the Euler parameters accordingly from equations (4.8) and (4.9).

$$\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \frac{|2\,q_0|}{2} \\ \frac{E_{32}-E_{23}}{2\,|2\,q_0|} \\ \frac{E_{13}-E_{31}}{2\,|2\,q_0|} \\ \frac{E_{21}-E_{12}}{2\,|2\,q_0|} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{E_{32}-E_{23}}{2\,|2\,q_1|} \\ \frac{|2\,q_1|}{2} \\ \frac{E_{12}+E_{21}}{2\,|2\,q_1|} \\ \frac{E_{13}+E_{31}}{2\,|2\,q_1|} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{E_{13}-E_{31}}{2\,|2\,q_2|} \\ \frac{E_{12}+E_{21}}{2\,|2\,q_2|} \\ \frac{|2\,q_2|}{2} \\ \frac{E_{23}+E_{32}}{2\,|2\,q_2|} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{E_{21}-E_{12}}{2\,|2\,q_3|} \\ \frac{E_{13}+E_{31}}{2\,|2\,q_3|} \\ \frac{E_{23}+E_{32}}{2\,|2\,q_3|} \\ \frac{|2\,q_3|}{2} \end{bmatrix} \tag{4.11}$$

The time derivatives of the Euler parameters equation (4.2), with the aid of equations (3.18) and (3.20), are for $\overrightarrow{\boldsymbol{\omega}}$ in terms of base $\overrightarrow{\underline{\boldsymbol{E}}}_r$

$$\begin{aligned} \dot{q}_0 &= -\tfrac{1}{2}\sin\tfrac{\vartheta}{2}\,\dot{\vartheta} & \dot{\overline{\boldsymbol{q}}} &= \tfrac{1}{2}\cos\tfrac{\vartheta}{2}\,\dot{\vartheta}\,\overline{\boldsymbol{a}} + \sin\tfrac{\vartheta}{2}\,\dot{\overline{\boldsymbol{a}}} \\ &= -\tfrac{1}{2}\sin\tfrac{\vartheta}{2}\,\underline{\boldsymbol{a}}\cdot\overline{\boldsymbol{\omega}}^r & &= \tfrac{1}{2}\cos\tfrac{\vartheta}{2}\,\overline{\boldsymbol{\omega}}^r + \tfrac{1}{2}\sin\tfrac{\vartheta}{2}\,\underline{\overline{\widetilde{\boldsymbol{a}}}}\cdot\overline{\boldsymbol{\omega}}^r \\ &= -\tfrac{1}{2}\,\underline{\boldsymbol{q}}\cdot\overline{\boldsymbol{\omega}}^r & &= \tfrac{1}{2}\,q_0\,\overline{\boldsymbol{\omega}}^r + \tfrac{1}{2}\,\underline{\overline{\widetilde{\boldsymbol{q}}}}\cdot\overline{\boldsymbol{\omega}}^r \end{aligned} \tag{4.12}$$

The same procedure can be repeated for $\overrightarrow{\boldsymbol{\omega}}$ in terms of base $\overrightarrow{\underline{\boldsymbol{E}}}_s$. Equation (4.12) can be rewritten in the more familiar matrix format

$$\begin{bmatrix} \dot{q}_0 \\ \dot{\overline{\boldsymbol{q}}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -\underline{\boldsymbol{\omega}}_r \\ \overline{\boldsymbol{\omega}}^r & -\underline{\overline{\widetilde{\boldsymbol{\omega}}}}_r^{\,r} \end{bmatrix}\cdot\begin{bmatrix} q_0 \\ \overline{\boldsymbol{q}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -\underline{\boldsymbol{\omega}}_s \\ \overline{\boldsymbol{\omega}}^s & +\underline{\overline{\widetilde{\boldsymbol{\omega}}}}_s^{\,s} \end{bmatrix}\cdot\begin{bmatrix} q_0 \\ \overline{\boldsymbol{q}} \end{bmatrix} \tag{4.13}$$

The constraint equation, equation (4.6) in differential form is

$$q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3 = \begin{bmatrix} q_0 & \underline{\boldsymbol{q}} \end{bmatrix}\cdot\begin{bmatrix} \dot{q}_0 \\ \dot{\overline{\boldsymbol{q}}} \end{bmatrix} = 0 \tag{4.14}$$

If equation (4.13) is substituted into equation (4.14), it confirms, as expected, that equation (4.13) still satisfies the constraint condition.

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